FUNCTIONS WITH ISOLATED SINGULARITIES ON SURFACES, II

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ABSTRACT. Let M be a smooth connected compact surface, P be either the real line $\mathbb{R}$ or the circle S^1. For a subset $X \subset M$ denote by $\mathcal{D}(M, X)$ the group of diffeomorphisms of M fixed on X. In this note we consider a special class $\mathcal{F}$ of smooth maps $f : M \to P$ with isolated singularities which includes all Morse maps. For each map $f \in \mathcal{F}$ we consider certain submanifolds $X \subset M$ that are "adopted" with f in a natural sense, and study the right action of the group $\mathcal{D}(M, X)$ on $C^\infty(M, P)$. The main result describes the homotopy types of the connected components of the stabilizers $\mathcal{S}(f)$ and orbits $\mathcal{O}(f)$ for all maps $f \in \mathcal{F}$. It extends previous author results on this topic.

1. Introduction

Let M be a smooth compact connected surface and P be either the real line $\mathbb{R}$ or the circle S^1. In this paper we study the subspace $\mathcal{F} \subset \mathcal{C}^\infty(M, P)$ consisting of maps $f : M \to P$ satisfying the following two axioms:

Axiom (B1). *The set Σ_f of critical points of f is finite and is contained in the interior* Int$M$, *and f takes a constant value at each boundary component of M.*

Axiom (L1). *For every critical point z of f there exists a local presentation $f_z : \mathbb{R}^2 \to \mathbb{R}$ of f in which $z = (0, 0)$ and f_z is a homogeneous polynomial without multiple factors.*

For instance, due to Morse lemma each non-degenerate critical point of a function $f : M \to P$ is equivalent to a homogeneous polynomial $\pm x^2 \pm y^2$ having no multiple factors. Hence each Morse function satisfies axiom (L1).

Recall that every homogeneous polynomial $g : \mathbb{R}^2 \to \mathbb{R}$ can be expressed as a product $g = L_1^{p_1} \cdots L_\alpha^{p_\alpha} Q_1^{q_1} \cdots Q_\beta^{q_\beta}$, where $L_i(x, y) = a_i x + b_i y$, and $Q_j(x, y) = c_j x^2 + 2d_j xy + e_j y^2$ is an irreducible over $\mathbb{R}$ (definite) quadratic form, $L_i/L_{i'} \neq$ const for $i \neq i'$, and $Q_j/Q_{j'} \neq$ const for $j \neq j'$. Then Axiom (L1) requires that $p_i = q_j = 1$ for all i, j.

Notice that if $p_i \geq 2$ for some i, then the line $\{L_i = 0\}$ consists of critical points of f, whence Axiom (L1) implies that all critical points of f are isolated. Moreover, the requirement that $q_j = 1$ for all j is a certain non-degeneracy assumption.

Definition 1.1. *Let $X \subset M$ be a compact submanifold such that its connected components may have distinct dimensions. Denote by X^i, $i = 0, 1, 2$, the union of connected components of X of dimension i. Let also $f : M \to P$ be a smooth map satisfying axiom* (B1). *We will say that X is an f-**adopted** if the following conditions hold true:*

(0) $X^0 \subset \Sigma_f$;
(1) $X^1 \cap \Sigma_f = \varnothing$ *and f takes constant value on each connected component of X^1;*
(2) *the restriction $f|_{X^2}$ satisfies axiom* (B1) *as well.*

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For instance, the following sets and their connected components are f-*adopted*: $\varnothing$, ∂M, Σ_f, $f^{-1}(c)$, where $c \in P$ is a regular value of f, $f^{-1}(I)$, where $I \subset P$ is a closed interval whose both ends are regular values of f.

Let $X \subset M$ be an f-adopted submanifold, and $\mathcal{D}(M,X)$ be the group of diffeomorphisms of M fixed on X. Endow $\mathcal{D}(M,X)$ and $\mathcal{C}^{\infty}(M,P)$ with $\mathcal{C}^{\infty}$-topologies. Then $\mathcal{D}(M,X)$ continuously acts from the right on $\mathcal{C}^{\infty}(M,P)$ by the formula:

$$f \cdot h = f \circ h, \qquad h \in \mathcal{D}(M,X), \ f \in \mathcal{C}^{\infty}(M,P). \tag{1.1}$$

For $f \in \mathcal{C}^{\infty}(M,P)$ let $\mathcal{S}(f,X) = \{h \in \mathcal{D}(M,X) \mid f \circ h = f\}$ and $\mathcal{O}(f,X) = \{f \circ h \mid h \in \mathcal{D}(M,X)\}$ be respectively the stabilizer and the orbit of f. Let also $\mathcal{D}_{\mathrm{id}}(M,X)$ and $\mathcal{S}_{\mathrm{id}}(f,X)$ be the identity path components of $\mathcal{D}(M,X)$ and $\mathcal{S}(f,X)$, and $\mathcal{O}_f(f,X)$ be the path component of f in $\mathcal{O}(f,X)$.

We will omit notation for X whenever it is empty, for instance $\mathcal{S}_{\mathrm{id}}(f) = \mathcal{S}_{\mathrm{id}}(f,\varnothing)$, and so on.

In a series of papers [6, 7, 9, 8] for the cases $X = \varnothing$ and $X = \Sigma_f$ the author calculated the homotopy types of $\mathcal{S}_{\mathrm{id}}(f,X)$ and $\mathcal{O}_f(f,X)$ for a large class of smooth maps $f : M \to P$ which includes all maps satisfying axioms (B1) and (L1).

The aim of this paper is to extend these results to the general case of $\mathcal{D}(M,X)$, where X is an f-adopted submanifold, see Section 2.

1.2. **Notation.** Throughout the paper T^2 will be a 2-torus $S^1 \times S^1$, $M\ddot{o}$ a Möbius band, and $\mathbb{K}$ a Klein bottle. For topological spaces X and Y the notation $X \cong Y$ will mean that X and Y are homotopy equivalent.

For a map $f : M \to P$ and $c \in P$ the set $f^{-1}(c)$ will be called a *level set* of f. Let ω be connected component of some level set of f. Then ω is *critical* if it contains a critical point of f. Otherwise ω will be called *regular*.

We will denote by Δ_f the partition on M whose elements are critical points of f and connected components of the sets $f^{-1}(c) \setminus \Sigma_f$ for all $c \in P$, see [6, 9].

For a vector field F on M and a smooth function $\alpha : M \to \mathbb{R}$ we will denote by $F(\alpha)$ the Lie derivative of α along F.

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2. Main results

In this section we will assume that $f : M \to P$ is a smooth map satisfying axioms (B1) and (L1), and $X \subset M$ is an f-adopted submanifold. The principal results of this paper are Theorems 2.1, 2.2, 2.6, and 2.7. They are new only for the case when X *is infinite.*

Homotopy type of $\mathcal{S}_{\mathrm{id}}(f,X)$. By [6, Th. 1.9] and [9, Th. 3] $\mathcal{S}_{\mathrm{id}}(f)$ is contractible except for the functions of types (A)-(D) of [6, Th. 1.9]:

(A) $f_A : S^2 \to P$ with only two non-degenerate critical points: one maximum and one minimum, and both points are non-degenerate.
(B) $f_B : D^2 \to P$ with a unique critical point being a non-degenerate local extreme;
(C) $f_C : S^1 \times I \to P$ without critical points.
(D) $f_D : T^2 \to S^1$ without critical points.

For these functions $\mathcal{S}_{\mathrm{id}}(f)$ is homotopy equivalent to S^1.

The following theorem describes the homotopy type of $\mathcal{S}_{\mathrm{id}}(f,X)$.

Theorem 2.1. c.f. [6, 9]. $\mathcal{S}_{\mathrm{id}}(f,X) \cong S^1$ *if and only if the following two conditions hold true*

(i) $\mathcal{S}_{\mathrm{id}}(f) \cong S^1$, *so* f *is of one of the types* (A)-(D) *above, and*

(ii) $X \subset \Sigma_f$.

In all other cases $\mathcal{S}_{\mathrm{id}}(f, X)$ *is contractible.*

The proof of this theorem will be given in Section 5.

Homotopy type of $\mathcal{O}_f(f, X)$. We will now show that for description of the homotopy type of orbits $\mathcal{O}_f(f, X)$ one can always assume that $\partial M \subset X$. First we need the following technical result.

Theorem 2.2. *The map* $p : \mathcal{D}(M, X) \to \mathcal{O}(f, X)$ *defined by* $p(h) = f \circ h$ *for* $h \in \mathcal{D}(M, X)$ *is a Serre fibration.*

For $X = \varnothing$ the orbit $\mathcal{O}_f(f)$ has a "finite codimension" in the space of all smooth functions and the result is proved in [11], see [9, Lm. 11] for detailed explanations. We will deduce the general case of Theorem 2.2 from the case $X = \varnothing$, see Section 6.

Corollary 2.3. *Let* Y *be a union of some connected components of* ∂M, *so* $X \cup Y$ *is an* f*-adopted submanifold of* M. *Then* $\mathcal{O}_f(f, X \cup Y) = \mathcal{O}_f(f, X)$.

Proof. We can assume that $X \cap Y = \varnothing$, otherwise just replace Y with $Y \setminus X$.

Evidently, $\mathcal{O}_f(f, X \cup Y) \subset \mathcal{O}_f(f, X)$.

Conversely, let $g \in \mathcal{O}_f(f, X)$, so there exists a path $\omega : I \to \mathcal{O}(f, X)$ such that $\omega_0 = f$, and $\omega_1 = g$. Since p is a Serre fibration, this path lifts to a path $\widetilde{\omega} : I \to \mathcal{D}(M, X)$ such that $\widetilde{\omega}_0 = \mathrm{id}_M$ and $\omega_t = f \circ \widetilde{\omega}_t$. In particular, $g = \omega_1 = f \circ \widetilde{\omega}_1$.

Since $\widetilde{\omega}_1$ is isotopic to id_M relatively to X, we have that it preserves each connected component of Y. Then due to (B1), it is easy to construct a diffeomorphism $h \in \mathcal{D}(M, X)$ such that $h = \widetilde{\omega}_1$ even on some neighbourhood of Y and $f \circ h = f$. Hence $h^{-1} \circ \widetilde{\omega}_1 \in \mathcal{D}(M, X \cup Y)$, and $g = f \circ \widetilde{\omega}_1 = f \circ h^{-1}\widetilde{\omega}_1 \in \mathcal{O}_f(f, X \cup Y)$. □

Lemma 2.4. [12, 1, 2, 3, 4]. *The homotopy types of* $\mathcal{D}_{\mathrm{id}}(M, X)$ *are presented in the following table:*

Case	(M, X)	*Homotopy type of* $\mathcal{D}_{\mathrm{id}}(M, X)$
1)	S^2, $\mathbb{R}\mathrm{P}^2$	$SO(3)$
2)	T^2	T^2
3)	$(S^2, *)$, $(S^2, **)$, $\mathbb{K}$	S^1
4)	$(D^2, *)$, D^2, $S^1 \times I$, *Mö*	
5)	*all other cases*	*point*

Here $*$ *is a point;* $(S^2, **)$ *means that* X *consists of two points; and we omit* X *when it is empty, e.g.* $S^2 = (S^2, \varnothing)$.

In particular, $\chi(M) < \#(X)$, *e.g., when* X *is infinite, then* $\mathcal{D}_{\mathrm{id}}(M, X)$ *is contractible.* □

Remark 2.5. *In the cases* 3) *and* 4) *the homotopy types of* $\mathcal{D}_{\mathrm{id}}(M, X)$ *are the same, but we separate these cases with respect to the existence of boundary. So in the case* 3) M *is closed while in the case* 4) $\partial M \neq \varnothing$.

Denote

$$\mathcal{S}'(f, X) := \mathcal{S}(f, X) \cap \mathcal{D}_{\mathrm{id}}(M, X).$$

Thus each $h \in \mathcal{S}'(f, X)$ preserves f and is isotopic to id_M, though that isotopy is not assumed to be f-preserving. This group plays an important role for the fundamental group $\pi_1\mathcal{O}_f(f, X)$. Notice that $\pi_0\mathcal{S}'(f, X)$ can be regarded as the kernel of the homomorphism:

$$i_0 : \pi_0\mathcal{S}(f, X) \to \pi_0\mathcal{D}(M, X).$$

induced by the inclusion $i : \mathcal{S}(f, X) \subset \mathcal{D}(M, X)$.

Theorem 2.6. *We have that*

$$\pi_n \mathcal{O}_f(f, X) = \pi_n \mathcal{D}_{\mathrm{id}}(M, X), \qquad n \geq 2. \tag{2.1}$$

Thus if $(M,X) = (S^2, \varnothing)$ *or* $(\mathbb{R}\mathrm{P}^2, \varnothing)$*, then* $\pi_n \mathcal{O}_f(f,X) = \pi_n S^2$*,* $n \geq 3$*, and* $\pi_2 \mathcal{O}_f(f,X) = 0$*. Otherwise,* $\pi_n \mathcal{O}_f(f,X) = 0$*,* $n \geq 2$*, i.e.* $\mathcal{O}_f(f,X)$ *is aspherical.*

Moreover, for $\pi_1 \mathcal{O}_f(f,X)$ *we have the following exact sequence*

$$0 \to \frac{\pi_1 \mathcal{D}_{\mathrm{id}}(M,X)}{\pi_1 \mathcal{S}_{\mathrm{id}}(f,X)} \xrightarrow{p_1} \pi_1 \mathcal{O}_f(f,X) \xrightarrow{\partial_1} \pi_0 \mathcal{S}'(f,X) \to 0. \tag{2.2}$$

In particular, in the case 5) *of Lemma* 2.4*, when* $\mathcal{D}_{\mathrm{id}}(M,X)$ *is contractible, we have an isomorphism*

$$\pi_1 \mathcal{O}_f(f,X) \approx \pi_0 \mathcal{S}'(f,X). \tag{2.3}$$

In the case 4) *denote* $Y = X \cup \partial M$*, then*

$$\pi_1 \mathcal{O}_f(f,X) = \pi_1 \mathcal{O}(f,Y) \approx \pi_0 \mathcal{S}'(f,Y). \tag{2.4}$$

Proof. (2.1). Suppose that $\mathcal{S}_{\mathrm{id}}(f,X)$ is contractible. Then from the exact sequence of homotopy groups of the fibration $p : \mathcal{D}(M,X) \to \mathcal{O}(f,X)$ we obtain that $\pi_n \mathcal{O}_f(f,X) = \pi_n \mathcal{D}_{\mathrm{id}}(M,X)$ for $n \geq 2$.

Now let $\mathcal{S}_{\mathrm{id}}(f,X) \cong S^1$. Then again $\pi_n \mathcal{O}_f(f,X) = \pi_n \mathcal{D}_{\mathrm{id}}(M,X)$ for $n \geq 3$, while for $n = 2$ we get the following part of exact sequence:

$$0 \to \pi_2 \mathcal{D}_{\mathrm{id}}(M,X) \xrightarrow{p_2} \pi_2 \mathcal{O}_f(f,X) \xrightarrow{\partial_2} \pi_1 \mathcal{S}(f,X) \xrightarrow{i_1} \pi_1 \mathcal{D}_{\mathrm{id}}(M,X)$$

In the proof of [6, Th. 1.9] it was shown that the map i_1 is a monomorphism, so $\pi_2 \mathcal{O}_f(f,X) = \pi_2 \mathcal{D}_{\mathrm{id}}(M,X)$ as well. Exact values of groups $\pi_n \mathcal{D}_{\mathrm{id}}(M,X)$ follow from Lemma 2.4.

(2.2). Since $\pi_2 \mathcal{O}_f(f,X) = 0$, we have the following exact sequence:

$$0 \to \pi_1 \mathcal{S}_{\mathrm{id}}(f,X) \xrightarrow{i_1} \pi_1 \mathcal{D}_{\mathrm{id}}(M,X) \xrightarrow{p_1} \pi_1 \mathcal{O}_f(f,X) \xrightarrow{\partial_1}$$
$$\xrightarrow{\partial_1} \pi_0 \mathcal{S}(f,X) \xrightarrow{i_0} \pi_0 \mathcal{D}(M,X),$$

which implies (2.2).

Finally (2.3) follows from (2.2), and (2.4) from (2.2) and Corollary 2.3. □

Fundamental group $\pi_1 \mathcal{O}_f(f,X)$**.** The following theorem shows that the computations of $\pi_1 \mathcal{O}_f(f,X)$ almost always reduces to the case when M is either D^2, or $S^1 \times I$, or $M\ddot{o}$. It extends [8, Th. 1.8] to the case when X is infinite.

Theorem 2.7. c.f. [8, Th. 1.8]. *Suppose one the following conditions holds true:*

(i) $\partial M \neq \varnothing$*;* (ii) $\chi(M) < 0$*;* (iii) X *is infinite.*

Then there exist finitely many f*-adapted mutually disjoint compact subsurfaces* $B_1, \ldots, B_n$ *with the following properties:*

- $\mathrm{Int} B_i \cap X \subset X^0$*;*
- *each* B_i *is diffeomorphic either to* D^2*, or* $S^1 \times I$*, or* $M\ddot{o}$*;*
- *put* $Y_i = \partial B_i \cup (B_i \cap X^0)$*, then*

$$\pi_1 \mathcal{O}_f(f,X) \approx \prod_{i=1}^{n} \pi_0 \mathcal{S}'(f|_{B_i}, Y_i). \tag{2.5}$$

The rest of the paper is devoted to proof of Theorems 2.1, 2.2, and 2.7.

3. Proof of Theorem 2.7

The proof will be given at the end of this section and now we will establish one technical result.

Let $f : S^1 \times I \to I$ be the function defined by $f(z,\tau) = \tau$. For a non-empty subset $A \subset I$ denote by $\mathcal{S}_A$ the stabilizer $\mathcal{S}\bigl(f, S^1 \times A\bigr)$, i.e. the group of diffeomorphisms h of $S^1 \times I$ such that

(1) $f \circ h = f$, so $h(S^1 \times \tau) = S^1 \times \tau$ for all τ;
(2) h is fixed on $S^1 \times A$.

Let $J \subset I = [0,1]$ be a *non-empty*, *closed*, and *connected* subset, T be a closed neighbourhood of J in I, and T' be a closed neighbourhood of T in I.

Lemma 3.1. *The inclusion $i : \mathcal{S}_T \subset \mathcal{S}_J$ is a homotopy equivalence, so there exists a homotopy*
$$H : \mathcal{S}_J \times [0,1] \longrightarrow \mathcal{S}_J,$$
such that $H_0 = \mathrm{id}(\mathcal{S}_J)$ and $H_1(\mathcal{S}_J) \subset \mathcal{S}_T$. Moreover, $H_s(h) = h$ on $S^1 \times (I \setminus T')$ for all $s \in [0,1]$ and $h \in \mathcal{S}_J$, and in particular, H_s is fixed on $\mathcal{S}_{T'}$.

Proof. Identify S^1 with the unit circle in the complex plane $\mathbb{C}$ and define the following vector field $F(z,\tau) = \frac{\partial}{\partial z}$ on $S^1 \times I$ generating the flow
$$\mathbf{F} : (S^1 \times I) \times \mathbb{R} \to S^1 \times I, \qquad \mathbf{F}(z,\tau,t) = (e^{2\pi i \tau} z, \tau).$$
We claim that there exists a unique map $\Delta : \mathcal{S}_J \to \mathcal{C}^\infty(S^1 \times I, \mathbb{R})$ continuous with respect to $\mathcal{C}^\infty$-topologies and such that

(a) $h(z,\tau) = \mathbf{F}\bigl(z,\tau,\Delta(h)(z,\tau)\bigr) = \Bigl(e^{2\pi i \cdot \Delta(h)(z,\tau)} z, \tau\Bigr)$.
(b) Let $Y \subset I$ be any closed connected subset containing J. Then $\Delta(h)(z,\tau) = 0$ for $(h,\tau) \in \mathcal{S}_Y \times Y$. In particular, $\Delta(h)(z,\tau) = 0$ for all $\tau \in J$.

Indeed, let $Q : \mathbb{R} \times I \to S^1 \times I$, $Q(t,\tau) = (e^{2\pi i t}, \tau)$ be the universal covering map of $S^1 \times I$. Then each $h \in \mathcal{S}_J$ lifts to a unique map
$$\tilde{h} = (\tilde{h}_1, \tilde{h}_2) : \mathbb{R} \times I \to \mathbb{R} \times I$$
such that $h \circ Q = Q \circ \tilde{h}$ and $\tilde{h}$ is fixed on $Q^{-1}(S^1 \times J)$. Put
$$\Delta(h)(t,\tau) = \tilde{h}_1(t,\tau) - t.$$
We claim that Δ satisfies conditions (a) and (b) above.

(a) Notice that
$$Q \circ \tilde{h}(t,\tau) = \left(e^{2\pi i \tilde{h}_1}, \tilde{h}_2\right), \qquad\qquad h \circ Q(t,\tau) = h(e^{2\pi i t}, \tau).$$
Then from the the identity $h \circ Q = Q \circ \tilde{h}$ we get
$$h(z,\tau) = h(e^{2\pi i t}, \tau) = \left(e^{2\pi i \tilde{h}_1}, \tau\right) = \left(e^{2\pi i t} \cdot e^{2\pi i [\tilde{h}_1(t,\tau) - t]}, \tau\right) =$$
$$= \left(e^{2\pi i \cdot \Delta(h)(t,\tau)} z, \tau\right) = \mathbf{F}\bigl(z,\tau,\Delta(h)(z,\tau)\bigr).$$

(b) Let $\tau \in Y$ and $h \in \mathcal{S}_Y \subset \mathcal{S}_J$. Since h is fixed on $S^1 \times Y$ and Y is connected, it follows that the lifting $\tilde{h}$ is fixed on $\mathbb{R} \times Y$, i.e. $\tilde{h}(t,\tau) = (t,\tau)$ for all $(t,\tau) \in \mathbb{R} \times Y$. This means that $\tilde{h}_1(t,\tau) = t$, whence $\Delta(h)(t,\tau) = t - t = 0$.

Now fix any $\mathcal{C}^\infty$-function $\mu : I \to [0,1]$ such that $\mu = 0$ on T and $\mu = 1$ on $\overline{I \setminus T'}$, and defined the homotopy $H : \mathcal{S}_J \times [0,1] \longrightarrow \mathcal{S}_J$, by
$$H(h,s) = \mathbf{F}\bigl(z,\tau,(s\mu(\tau) + 1 - s) \cdot \Delta(h)(z,\tau)\bigr).$$

We claim that H satisfies statement of lemma.

1) First notice that $H_0 = \mathrm{id}$. Indeed,

$$H(h,0)(z,\tau) = \mathbf{F}\bigl(z,\tau,\Delta(h)(z,\tau)\bigr) \xlongequal{(a)} h(z,\tau).$$

2) $H(h,s)$ is fixed on $S^1 \times J$. Indeed, if $(\tau \in J$, then $\Delta(h)(z,\tau) = 0$, whence

$$H(h,s)(z,\tau) = \mathbf{F}\bigl(z,\tau,0\bigr) = (z,\tau).$$

3) Let us verify that $H(h,s)$ is a diffeomorphism. Notice $H(h,s)$ is obtained by substitution of a smooth function $\alpha = (s\mu + 1 - s)\cdot\Delta(h)$ into the flow map instead of time. Then, due to [5], $H(h,s)$ is a diffeomorphism if and only if the Lie derivative

$$F(\alpha) > -1. \tag{3.1}$$

In particular, since $h = H(h,0)$ is a diffeomorphism we have that $F(\Delta(h)) > -1$. Hence

$$F\bigl((s\mu+1-s)\cdot\Delta(h)\bigr) = F(s\mu+1-s)\cdot\Delta(h) + (s\mu+1-s)F\bigl(\Delta(h)\bigr).$$

Since μ depends only on τ, we obtain that $F(s\mu+1-s) = 0$, and so the first summand vanishes. Moreover, $0 \le s\mu+1-s \le 1$, whence we get the inequality:

$$F\bigl((s\mu+1-s)\cdot\Delta(h)\bigr) = (s\mu+1-s)F\bigl(\Delta(h)\bigr) > -1,$$

as well. Thus $H(h,s)$ is a diffeomorphism.

4) Since $f \circ \mathbf{F}(z,\tau,t) = f(z,\tau)$, it follows that $f \circ H(h,s) = f$ for all $(h,s) \in \mathcal{S}_J \times I$. Thus $H(h,s) \in \mathcal{S}_J$.

5) Let us show that $H_1(\mathcal{S}_J) \subset \mathcal{S}_T$, i.e. $H(h,1)$ is fixed on $S^1 \times T$. Let $\tau \in T$. Then $\mu(\tau) = 0$, whence

$$H(h,1)(z,\tau) = \mathbf{F}\bigl(z,\tau,(1\cdot\mu(\tau)+1-1)\cdot\Delta(h)(z,\tau)\bigr) = \mathbf{F}(z,\tau,0) = (z,\tau).$$

6) Finally, let us verify that $H(h,s) = h$ on $S^1 \times (I \setminus T')$. Let $\tau \in I \setminus T'$. Then $\mu(\tau) = 1$, whence

$$H(h,s)(z,\tau) = \mathbf{F}\bigl(z,\tau,(s\mu(\tau)+1-s)\cdot\Delta(h)(z,\tau)\bigr) = \mathbf{F}\bigl(z,\tau,\Delta(h)(z,\tau)\bigr) = h(z,\tau).$$

Lemma is proved. □

Remark 3.2. *Notice that the map $H_1 : \mathcal{S}_J \to \mathcal{S}_T$ is not a retraction.*

Let X be an f-adopted submanifold with $X^0 = \varnothing$ and N be a neighbourhood of X. For every connected component Y of X let N_Y be the connected component of N containing Y.

Definition 3.3. *Say that N is **f-adopted** if it has the following properties.*

(1) *$\overline{N_Y} \cap \overline{N_{Y'}} = \varnothing$ for any pair of distinct components Y, Y' of X.*

(2) *Let Y be a connected component of X^1. Put $J = [0,1]$ if Y is a boundary component of M, and $J = [-1,1]$ otherwise. Then there exists a diffeomorphism $q : S^1 \times J \to N_Y$ such that $q(S^1 \times 0) = Y$, for each $t \in J$ the set $q(S^1 \times t)$ is a regular component some level set of f.*

(3) *Let Y be a connected component of X^2, and $\gamma_1, \dots, \gamma_n$ be the set of all boundary components of ∂Y that belong to the interior $\mathrm{Int}M$. Then N_Y is obtained from Y by gluing collars $C_i = S^1 \times [0,1]$ to each of γ_i along $S^1 \times 0$, so that for every $t \in [0,1]$ the set $S^1 \times t$ corresponds to some level set of f.*

As a consequence of Lemma 3.1 we get the following statement.

Corollary 3.4. *Let $X \subset M$ be an f-adopted submanifold, and $\hat{N}$ be an f-adopted neighbourhood of $X^1 \cup X^2$. Denote $\hat{X} = X^0 \cup \hat{N}$. Then the inclusion*

$$\mathcal{S}(f, \hat{X}) \cap \mathcal{D}(M, \hat{X}) \ \subset \ \mathcal{S}(f, X) \cap \mathcal{D}(M, X)$$

is a homotopy equivalence. In particular, so is the inclusion $\mathcal{S}'(f, \hat{X}) \subset \mathcal{S}'(f, X)$.

Proof. Let $h \in \mathcal{S}(f, X) \cap \mathcal{D}(M, X)$. We should construct a "canonical" deformation of h in $\mathcal{S}(f, X)$ to a diffeomorphism fixed on a neighbourhood $\hat{N}$ on $X^1 \cup X^2$, and this deformation should be supported in some neighbourhood of $X^1 \cup \partial X^2$.

Let Y be a connected component of $X^1 \cup \partial X^2$. Then Y has a neighbourhood U diffeomorphic to the cylinder $S^1 \times I$ such that each set $S^1 \times \tau$ is a regular component of some level set of f. Then by Lemma 3.1 there exists a deformation of $\mathcal{S}(f|_U, U \cap X)$ into $\mathcal{S}(f|_U, U \cap \hat{N})$ with supports in $\mathrm{Int}U$.

Applying this to each connected component of $X^1 \cup \partial X^2$ we will get a deformation retraction of $\mathcal{S}(f, X) \cap \mathcal{D}(M, X)$ onto $\mathcal{S}(f, \hat{X}) \cap \mathcal{D}(M, \hat{X})$. The details are left to the reader. $\square$

Corollary 3.5. *Suppose $X^1 \cup X^2 \neq \varnothing$. Let $M_1, \dots, M_n$ be the closures of the connected components of $M \setminus (X^1 \cup X^2)$, and $Y_i = M_i \cap X$. Then we have the following commutative diagram consisting of isomorphisms:*

$$\begin{array}{ccc} \pi_1 \mathcal{O}_f(f, X) & \xrightarrow{\ \mu\ } & \prod_{i=1}^n \pi_1 \mathcal{O}(f|_{M_i}, Y_i) \\ {\scriptstyle \partial_1} \big\downarrow & & \big\downarrow {\scriptstyle \prod_{i=1}^n (\partial_1)_i} \\ \pi_0 \mathcal{S}'(f, X) & \xrightarrow{\ \eta\ } & \prod_{i=1}^n \pi_0 \mathcal{S}'(f|_{M_i}, Y_i). \end{array} \tag{3.2}$$

for some isomorphisms μ and η, where $(\partial_1)_i : \pi_1 \mathcal{O}(f|_{M_i}, Y_i) \to \pi_0 \mathcal{S}'(f|_{M_i}, Y_i)$ is the boundary homomorphism.

Proof. Since X and each Y_i is infinite, it follows from (2.3) that ∂_1 and $(\partial_1)_i$ are isomorphisms, and so are the vertical arrows. It is sufficient to define η, then μ will be uniquely determined.

Let $\hat{N}$ be an f-adopted neighbourhood of $X^1 \cup X^2$ and $\hat{X} = X^0 \cup \hat{N}$. Denote $\hat{Y}_i = M_i \cap \hat{X}$, $i = 1, \dots, n$. Then by Corollary 3.4 we have isomorphisms $\pi_0 \mathcal{S}'(f, X) \approx \pi_0 \mathcal{S}'(f, \hat{X})$, and $\pi_0 \mathcal{S}'(f|_{M_i}, Y_i) \approx \pi_0 \mathcal{S}'(f|_{M_i}, \hat{Y}_i)$.

Notice that the following map

$$\eta' : \mathcal{S}'(f, \hat{X}) \longrightarrow \prod_{i=1}^n \mathcal{S}'(f|_{M_i}, \hat{Y}_i), \qquad \eta'(h) = \big(h|_{M_1}, \dots, h|_{M_n}\big)$$

is a group isomorphism, since the restrictions $h|_{M_i}$, $i = 1, \dots, n$, have disjoint supports. Therefore η' it induces an isomorphism η from (3.2). $\square$

Proof of Theorem 2.7. Consider the following cases. (a) $\chi(M) < 0$ *and X is finite.* For $X = \varnothing$ the result was proved in [8, Th. 1.8]. However, the analysis of the proof shows that the same arguments hold for $X \subset \Sigma_f$ as well.

(b) $\chi(M) < 0$, $\varnothing \neq \partial M \subset X \subset \partial M \cup \Sigma_f$. By Corollary 2.3 we have that $\mathcal{O}_f(f, X) = \mathcal{O}(f, X^0)$, whence the decomposition from (a) holds in this case.

(c) X *is infinite.* Again by Corollary 2.3 we can assume that $\partial M \subset X$. Then by Corollary 3.5 we can write

$$\pi_1 \mathcal{O}_f(f, X) \approx \prod_{i=1}^n \pi_1 \mathcal{O}(f|_{M_i}, Y_i),$$

where $M_1, \dots, M_n$ are the closures of the connected components of $M \setminus (X^1 \cup X^2)$, and $Y_i = \partial M_i \cup (M_i \cap X^0) \neq \varnothing$. If $\chi(M_i) \geq 0$, then M_i is either a D^2, $S^1 \times I$, or $M\ddot{o}$, and so it satisfies the statement of theorem. Otherwise, $\chi(M_i) < 0$, and we can decompose $\pi_1 \mathcal{O}(f|_{M_i}, Y_i)$ by the case (b).

Evidently, (a)-(c) include all the cases (i)-(iii). Theorem is completed.

4. Axioms for a map $f : M \to P$

In this section we will present additional three axioms (B2)-(B4) for a smooth map $f : M \to P$ satisfying axiom (B1). These axioms are consequences of (L1). In the last two sections we will prove Theorems 2.1 and 2.2 for maps $f : M \to P$ satisfying (B1)-(B4).

First we introduce some notation. For a vector field F on M tangent to ∂M denote by $\mathbf{F} : M \times \mathbb{R} \to M$ the flow of F, and by $\varphi : \mathcal{C}^\infty(M, P)$ the *shift map* of F defined by

$$\varphi(\alpha)(x) = \mathbf{F}(x, \alpha(x))$$

for $\alpha \in \mathcal{C}^\infty(M, P)$ and $x \in M$.

Say that a vector-field F on M is *skew-gradient* with respect to f, if $F(f) \equiv 0$, and $F(z) = 0$ if and only if z is a critical point of f. In particular, f is constant along orbits of F.

Let M be a non-orientable compact surface. Then we will always denote by $\beta : \widetilde{M} \to M$ the orientable double covering of M and by ξ the orientation reversing involution of $\widetilde{M}$ which generates the group $\mathbb{Z}_2$ of covering transformations of $\widetilde{M}$.

Moreover, for a $\mathcal{C}^\infty$-map $f : M \to P$ we put $\widetilde{f} = \beta \circ f : \widetilde{M} \to P$, and denote by $\widetilde{\mathcal{D}}(\widetilde{M})$ the group of diffeomorphisms h of $\widetilde{M}$ commuting with ξ, i.e. $\tilde{h} \circ \xi = \xi \circ \tilde{h}$. Let also $\widetilde{\mathcal{S}}(\widetilde{f}) = \{\tilde{h} \in \widetilde{\mathcal{D}}(\widetilde{M}) \mid \widetilde{f} \circ \tilde{h} = \widetilde{f}\}$ be the stabilizer of $\widetilde{f}$ with respect to the right action of the group $\widetilde{\mathcal{D}}(\widetilde{M})$, and $\widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f})$ be the identity path component of $\widetilde{\mathcal{S}}(\widetilde{f})$.

Notice that each $\tilde{h} \in \widetilde{\mathcal{D}}_{\mathrm{id}}(\widetilde{M})$ induces a unique diffeomorphism h of M, and the correspondence $\tilde{h} \mapsto h$ is a homeomorphism $\nu : \widetilde{\mathcal{D}}_{\mathrm{id}}(\widetilde{M}) \to \mathcal{D}_{\mathrm{id}}(M)$, which induces a homeomorphism $\nu : \widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f}) \to \mathcal{S}_{\mathrm{id}}(f)$.

Axiom (B2)**.** *Suppose M is orientable. Then there exists a **skew-gradient** with respect to f vector field F on M satisfying the following conditions.*

- *Consider the following **convex** subset of $\mathcal{C}^\infty(M, \mathbb{R})$:*

  $$\Gamma = \{\alpha \in \mathcal{C}^\infty(M, \mathbb{R}) \mid F(\alpha) > -1\}.$$

  Then $\varphi(\Gamma) = \mathcal{S}_{\mathrm{id}}(f)$.
- *If f has a critical point which is either non-extremal or degenerate extremal, then $\varphi|_\Gamma : \Gamma \to \mathcal{S}_{\mathrm{id}}(f)$ is a homeomorphism with respect to $\mathcal{C}^\infty$-topologies, and so $\mathcal{S}_{\mathrm{id}}(f)$ is contractible.*

  *Otherwise, $\varphi|_\Gamma : \Gamma \to \mathcal{S}_{\mathrm{id}}(f)$ is a $\mathbb{Z}$-covering map, and $\mathcal{S}_{\mathrm{id}}(f) \cong S^1$. In this case there is a **strictly positive** function $\theta \in \Gamma$ such that for any $\alpha, \beta \in \Gamma$ we have that $\varphi(\alpha) = \varphi(\beta)$ if and only if $\alpha - \beta = n\theta$ for some $n \in \mathbb{Z}$.*

*If M is non-orientable, then there exists a **skew-gradient** with respect to $\widetilde{f}$ vector field F on $\widetilde{M}$ satisfying the following conditions.*

- *F is skew-symmetric with respect to ξ, in the sense that $\xi^* F = -F$, which is equivalent to the assumption that $\mathbf{F}_\theta \circ \xi = \xi \circ \mathbf{F}_{-\theta}$ for all $\theta \in \mathbb{R}$.*
- *Define the following **convex** subset $\mathcal{C}^\infty(\widetilde{M}, \mathbb{R})$:*

  $$\widetilde{\Gamma} = \{\alpha \in \mathcal{C}^\infty(\widetilde{M}, \mathbb{R}) \mid F(\alpha) > -1, \ \alpha \circ \xi = -\alpha\}.$$

Then $\varphi(\widetilde{\Gamma}) = \widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f})$, and the restriction of shift map $\varphi : \widetilde{\Gamma} \to \widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f})$ is a homeomorphism with respect to $\mathcal{C}^\infty$-topologies, whence $\widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f})$ and $\mathcal{S}_{\mathrm{id}}(f) = \nu\big(\widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f})\big)$ are contractible.

Axiom (B3). *The map $p : \mathcal{D}(M) \to \mathcal{O}(f)$ defined by $p(h) = f \circ h$ for $h \in \mathcal{D}(M)$ is a Serre fibration.*

Axiom (B4). *Let $Y \subset M$ be a subsurface such that $f|_Y$ satisfies* (B1). *If f also satisfies* (B2) *and* (B3)*, then so does $f|_Y$.*

The following lemma summarizes certain results obtained in [6, 9].

Lemma 4.1. [9] *Axioms* (B1) *and* (L1) *imply all other axioms* (B2)-(B4).

Proof. In the paper [9] the author introduced three axioms (A1)-(A3) for a smooth map $f : M \to P$ such that (B1)=(A1), (B3)=(A3), (B1)&(L1)⇒(A1)-(A3) by [9, Lm. 12], and (A1)-(A3)⇒(B2) by [9, Th. 3]. To verify (B4), suppose $Y \subset M$ is a submanifold such that $f|_Y$ satisfies (B1). Then $f|_Y$ satisfies (L1), and therefore all other axioms hold true. □

5. Proof of Theorem 2.1

The orientable case of Theorem 2.1 is contained in the following lemma:

Lemma 5.1. *Suppose M is orientable, and $f : M \to P$ satisfies* (B1) *and* (B2)*. Let F, $\mathbf{F}$, φ, and Γ be the same as in* (B2)*. Denote $\Gamma_X = \{\alpha \in \Gamma \mid \alpha|_{X^1 \cup X^2} = 0\}$. Then*

$$\varphi(\Gamma_X) = \mathcal{S}_{\mathrm{id}}(f, X). \tag{5.1}$$

Moreover, $\mathcal{S}_{\mathrm{id}}(f, X) \cong S^1$ iff $\mathcal{S}_{\mathrm{id}}(f) \cong S^1$ and $X \subset \Sigma_f$. Otherwise $\mathcal{S}_{\mathrm{id}}(f, X)$ is contractible.

Proof. (5.1). Let $\alpha \in \Gamma_X$, i.e. $\alpha(x) = 0$ for all $x \in X^1 \cup X^2$. Then $\varphi(\alpha)$ is fixed on X, i.e. $\varphi(\alpha) \in \mathcal{S}_{\mathrm{id}}(f, X)$.

Indeed, if $x \in X^1 \cup X^2$, then $\varphi(\alpha)(x) = \mathbf{F}(x, \alpha(x)) = \mathbf{F}(x, 0) = x$. Moreover, every $x \in X^0$ is a critical point of f, and so $\mathbf{F}(x, t) = x$ for all $t \in \mathbb{R}$. In particular, $\varphi(\alpha)(x) = \mathbf{F}(x, \alpha(x)) = x$, and so $\varphi(\alpha) \in \mathcal{S}(f, X)$.

Since Γ is connected, $\varphi(0) = \mathrm{id}_M \in \mathcal{S}(f, X)$, and φ is continuous, we get that $\varphi(\alpha) \in \mathcal{S}_{\mathrm{id}}(f, X)$.

Conversely, suppose $h \in \mathcal{S}_{\mathrm{id}}(f, X)$, so we have an isotopy $h_t : M \to M$ in $\mathcal{S}_{\mathrm{id}}(f, X)$ between $h_0 = \mathrm{id}_M$ and $h_1 = h$. Since φ induces a covering map of Γ onto $\mathcal{S}_{\mathrm{id}}(f)$, we can lift the homotopy h_t (regarded as a continuous path in $\mathcal{S}_{\mathrm{id}}(f, X) \subset \mathcal{S}_{\mathrm{id}}(f)$) to Γ and get a homotopy of functions $\alpha_t : M \to \mathbb{R}$, $t \in [0, 1]$, such that $h_t(x) = \mathbf{F}(x, \alpha_t(x))$. Moreover, as $h_0 = \mathrm{id}_M$ we can assume that $\alpha_0 \equiv 0$. We claim that $\alpha_t \in \Gamma_X$ i.e. $\alpha_t = 0$ on $X^1 \cup X^2$.

For each point $x \in X^1 \cup X^2$ consider the set $\Lambda_x = \{\tau \in \mathbb{R} \mid \mathbf{F}(x, \tau) = x\}$ of periods of x, so $\alpha_t(x) \in \Lambda_x$ for all $t \in [0, 1]$. Notice that $\Lambda_x = \{0\}$ if x is non-periodic point, $\Lambda_x = \theta_x\mathbb{Z}$ for a periodic point of period θ_x, and $\Lambda_x = \mathbb{R}$ if x is a fixed point, i.e. a critical point of f.

Since for every non-fixed point x the set Λ_x is discrete and contains 0, and $\alpha_0 = 0$, it follows that $\alpha_t = 0$ on $(X^1 \cup X^2) \setminus \Sigma_f$ for all $t \in \mathbb{R}$. But Σ_f is nowhere dense, therefore $\alpha_t = 0$ on all of $X^1 \cup X^2$, i.e. $\alpha_t \in \Gamma_X$. This proves (5.1).

Now we can describe the homotopy type of $\mathcal{S}_{\mathrm{id}}(f, X)$. Notice that Γ and Γ_X are convex subsets of $\mathcal{C}^\infty(M, \mathbb{R})$ and therefore contractible.

1) If $\mathcal{S}_{\mathrm{id}}(f, X)$ is contractible, i.e. $\varphi : \Gamma \to \mathcal{S}_{\mathrm{id}}(f)$ is a homeomorphism, then $\varphi : \Gamma_X \to \mathcal{S}_{\mathrm{id}}(f, X)$ is a homeomorphism as well, whence $\mathcal{S}_{\mathrm{id}}(f, X)$ is also contractible.

2) Suppose $\mathcal{S}_{\mathrm{id}}(f,X) \cong S^1$, so $\varphi : \Gamma \to \mathcal{S}_{\mathrm{id}}(f)$ is a $\mathbb{Z}$-covering map. Consider two cases.

a) Suppose $X^1 \cup X^2 = \varnothing$, and so $X = X^0 \subset \Sigma_f$. In this case $\Gamma_X = \Gamma$, whence $\mathcal{S}_{\mathrm{id}}(f,X) = \mathcal{S}_{\mathrm{id}}(f) \cong S^1$.

b) Let $X^1 \cup X^2 \neq \varnothing$. Then the restriction map $\varphi|_{\Gamma_X} : \Gamma_X \to \mathcal{S}_{\mathrm{id}}(f,X)$ is injective. Hence due to (5.1) it is a homeomorphism onto.

Indeed, suppose $\varphi(\alpha) = \varphi(\beta)$ for some $\alpha, \beta \in \Gamma_X$. Then by (B2), $\alpha = \beta + n\theta$ for some $n \in \mathbb{Z}$. However, $\theta > 0$ on all M, while $\alpha = \beta = 0$ on $X^1 \cup X^2$. Therefore $n = 0$ and so $\alpha = \beta$ of all of M. □

Suppose M is non-orientable. Then $\widetilde{X} = \beta^{-1}(X)$ is $\widetilde{f}$-adopted submanifold of $\widetilde{M}$. Let $\widetilde{\mathcal{S}}(\widetilde{f}, \widetilde{X})$ be the subgroup of $\mathcal{S}(\widetilde{f}, \widetilde{X})$ consisting of diffeomorphisms commuting with ξ and $\widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f}, \widetilde{X})$ be its identity path-component. By the arguments similar to the proof of Lemma 5.1 one can show that $\widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f}, \widetilde{X})$ is contractible. Moreover, the homeomorphism $\nu : \widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f}) \to \mathcal{S}_{\mathrm{id}}(f)$ maps $\widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f}, \widetilde{X})$ onto $\mathcal{S}_{\mathrm{id}}(f,X)$, whence $\mathcal{S}_{\mathrm{id}}(f,X)$ is contractible as well.

6. Proof of Theorem 2.2

Theorem 2.2 is contained in the following theorem.

Theorem 6.1. *Suppose $f : M \to P$ satisfies axioms* (B1)-(B4). *Then the restriction map $p|_{\mathcal{D}(M,X)} : \mathcal{D}(M,X) \to \mathcal{O}(f,X)$ is a Serre fibration as well.*

Proof. Let S be a finite path-connected CW-complex and $s_0 \in S$ be a point. Let also $\psi : S \times I \to \mathcal{O}(f,X)$ be a homotopy such that $\psi(s_0, 0) = f$, and the restriction $\psi|_{S\times 0} : S \times 0 \to \mathcal{O}(f,X)$ lifts to a map $\eta_0 : S \to \mathcal{D}(M,X)$ satisfying $\eta_0(s_0) = \mathrm{id}_M$ and $\psi(s,0) = p(\eta_0(s)) = f \circ \eta_0(s)$. In particular, $\eta_0(s)|_X = \mathrm{id}_X$ for all $s \in S$.

We will prove that η_0 extends to a map $\eta : S \times I \to \mathcal{D}(M,X)$ such that $\psi = p \circ \eta$.

Since $p : \mathcal{D}(M) \to \mathcal{O}(f)$ is a Serre fibration, it follows that η_0 extends to a map $\kappa : S \times I \to \mathcal{D}(M)$ such that $\psi = p \circ \kappa$. So we have the following commutative diagram:

$$\begin{tikzcd} S \times 0 \arrow[r, "\eta_0"] \arrow[d, hook] & \mathcal{D}(M,X) \arrow[r, hook] & \mathcal{D}(M) \arrow[d, "p"] \\ S \times I \arrow[r, "\psi"'] \arrow[ur, dashed, "\eta"] \arrow[urr, "\kappa"] & \mathcal{O}(f,X) \arrow[r, hook] & \mathcal{O}(f) \end{tikzcd}$$

Notice also that κ induces a continuous map

$$K : S \times I \times M \to M, \qquad K(s,t,x) = \kappa(s,t)(x).$$

Lemma 6.2. *Let $(s,t) \in S \times I$ and γ be a leaf of Δ_f contained in X. Then $\kappa(s,t)(\gamma) = \gamma$. If γ is 1-dimensional, then $\kappa(s,t)$ also preserves orientation of γ.*

Proof. By definition $\psi(s,t) = f \circ \kappa(s,t)$. On the other hand, as $\psi(s,t) \in \mathcal{O}(f,X)$, there exists $\lambda \in \mathcal{D}(M,X)$ which depends on (s,t) and such that $\psi(s,t) = f \circ \lambda$ as well. Hence $f \circ \kappa(s,t) \circ \lambda^{-1} = f$, and so

$$\kappa(s,t) \circ \lambda^{-1}(\Sigma_f) = \Sigma_f, \qquad \qquad \kappa(s,t) \circ \lambda^{-1}(f^{-1}(c)) = f^{-1}(c)$$

for all $c \in P$. Moreover, as λ is fixed on X, we obtain that

$$\kappa(s,t)\bigl(\Sigma_f \cap X\bigr) \subset \Sigma_f, \qquad \qquad \kappa(s,t)\bigl(f^{-1}(c) \cap X\bigr) \subset f^{-1}(c),$$

for all $(s,t) \in S \times I$, that is

$$K\Bigl(S \times I \times [\Sigma_f \cap X]\Bigr) \subset \Sigma_f, \qquad \qquad K\Bigl(S \times I \times [f^{-1}(c) \cap X]\Bigr) \subset f^{-1}(c).$$

Now notice that there are three possibilities for γ:

(i) γ is a critical point of f,
(ii) γ is a regular component of some level set $f^{-1}(c)$, $c \in P$,
(iii) there is a critical component ω of some level set $f^{-1}(c)$ such that γ is a connected component of $\omega \setminus \Sigma_f$.

Let z be a critical point of f and ω be a connected component of $f^{-1}(c) \cap X$. Since $\kappa(s_0, 0) = \mathrm{id}_M$ and the sets $S \times I \times \{z\}$ and $S \times I \times \omega$ are connected, we obtain that $K\big(S \times I \times \{z\}\big) = \{z\}$, and $K\big(S \times I \times \omega\big) = \omega$. Hence $K\big(S \times I \times \alpha\big) = \alpha$ for any connected component α of $\omega \setminus \Sigma_f$. In other words,

$$\text{(i) } \kappa(s,t)(z) = z, \qquad \text{(ii) } \kappa(s,t)(\omega) = \omega, \qquad \text{(iii) } \kappa(s,t)(\alpha) = \alpha,$$

and, moreover, $\kappa(s,t)$ preserves orientation of ω and α because $\kappa(s_0, 0)$ does so. This proves our lemma for all the cases (i)-(iii) of γ. $\square$

The lemma says, in particular, that the lifting κ fixes X^0. We will find the lifting which also fixes $X^1 \cup X^2$.

Choose an f-adopted neighbourhood of $X^1 \cup X^2$. Let also Y be a connected component of $X^1 \cup X^2$. We will now distinguish three cases of Y:

(A) Y is an orientable surface,
(B) Y is a non-orientable surface,
(C) Y is a regular component of some level set of f, so Y is a circle.

Lemma 6.3. *Suppose Y belongs to the cases* (A) *or* (C). *Let also F, $\mathbf{F}$, φ, and Γ be the same as in Axiom* (B2). *Then there exists a continuous map $\hat{\delta} : S \times I \to \Gamma \subset \mathcal{C}^\infty(M, \mathbb{R})$ having the following properties:*

(a) *$\hat{\delta}(s, 0) = 0$ of M for all $s \in S$;*
(b) *$\mathrm{supp}\,\big(\hat{\delta}(s,t)\big) \subset N_Y$ for each $(s,t) \in S \times I$;*
(c) *Define the map $\hat{\kappa} = \varphi \circ \hat{\delta} : S \times I \xrightarrow{\hat{\delta}} \Gamma \xrightarrow{\varphi} \mathcal{S}_{\mathrm{id}}(f)$, so*
$$\hat{\kappa}(s,t)(x) = \mathbf{F}\big(x, \hat{\delta}(s,t)(x)\big).$$
Then $\hat{\kappa}(s,t) = \kappa(s,t)$ on Y for each $(s,t) \in S \times I$.

Hence the map $\eta : S \times I \to \mathcal{D}(M, X)$ defined by
$$\eta(s,t) = \hat{\kappa}(s,t)^{-1} \circ \kappa(s,t)$$
is a required lifting of ψ.

Proof. **Case (A).** Now Y is an *orientable* component of X^2. Then by (B4) the restriction of f to Y satisfies axiom (B2). Let
$$\Gamma_Y = \{\alpha \in \mathcal{C}^\infty(Y, \mathbb{R}) \mid F(\alpha) > -1\}$$
and $\varphi_Y : \Gamma_Y \to \mathcal{S}_{\mathrm{id}}(f|_Y)$, $\varphi_Y(\alpha)(x) = \mathbf{F}(x, \alpha(x))$, be the corresponding covering map described in (2) of (B2).

Due to Lemma 6.2 $f \circ \kappa(s,t)(x) = f(x)$ for $(s,t,x) \in S \times I \times Y$. In other words, the restriction $\kappa(s,t)|_Y$ of $\kappa(s,t)$ to Y belongs to the stabilizer $\mathcal{S}(f|_Y)$ of the restriction $f|_Y$ with respect to the right action of the group $\mathcal{D}(Y)$. Moreover, since $\kappa(s,0) = \mathrm{id}_M$ and $S \times I \times Y$ is connected, it follows that $\kappa(s,t)|_Y \in \mathcal{S}_{\mathrm{id}}(f|_Y)$.

Consider the restriction to Y map:
$$\kappa_Y : S \times I \to \mathcal{S}_{\mathrm{id}}(f|_Y), \qquad \kappa_Y(s,t)(x) = \kappa(s,t)(x)$$
for $(s,t,x) \in S \times I \times M$. This map is continuous into $\mathcal{C}^\infty$-topology of $\mathcal{S}_{\mathrm{id}}(f|_Y)$.

Since $\varphi_Y|_{\Gamma_Y}$ is a covering map, and $\kappa_Y(s,0) = \mathrm{id}_Y$ for all $s \in S$, we can lift $\kappa_Y|_{S \times 0}$ to a map $\delta : S \times 0 \to \Gamma_Y$ by the formula $\delta(s,0) = 0 : Y \to \mathbb{R}$ for all $s \in S$. Then from covering

homotopy property of $\varphi_Y|_{\Gamma_Y}$ we get that κ_Y extends to a lift $\delta : S \times I \to \Gamma_Y$ such that the following diagram is commutative:

$$
\begin{tikzcd}
 & \Gamma_Y \arrow[r, hook] \arrow[d, "\varphi_Y"] & \mathcal{C}^\infty(Y,\mathbb{R}) \\
S \times I \arrow[ur, "\delta"] \arrow[r, "\kappa_Y"] & \mathcal{S}_{\mathrm{id}}(f|_Y) \arrow[r, hook] & \mathcal{D}(Y)
\end{tikzcd}
\tag{6.1}
$$

In other words,

$$\kappa(s,t)(x) = \kappa_Y(s,t)(x) = \mathbf{F}\bigl(x, \delta(s,t)(x)\bigr) \tag{6.2}$$

for $x \in Y$.

Notice that for a neighbourhood N_Y there exists an *linear* extension operator

$$E : \mathcal{C}^\infty(Y,\mathbb{R}) \longrightarrow \mathcal{C}^\infty(M,\mathbb{R})$$

such that $E\alpha|_Y = \alpha$, and $\operatorname{supp}(E\alpha) \subset N_Y$ for each $\alpha \in \mathcal{C}^\infty(Y,\mathbb{R})$, see [10]. Define the composition

$$\delta' = E \circ \delta : S \times I \longrightarrow \mathcal{C}^\infty(M,\mathbb{R})$$

and consider the following map

$$K' : S \times I \times M \to M, \qquad K'(s,t,x) = \mathbf{F}\bigl(x, \delta'(s,t)(x)\bigr).$$

Evidently, $K'(s,t,x) = K(s,t,x)$ for $x \in Y$, so the restriction $K_{s,t}$ to Y is a diffeomorphism, and therefore we have the following inequality, see (3.1):

$$F(\delta'(s,t))(x) > -1, \qquad x \in Y. \tag{6.3}$$

As $S \times I \times Y$ is compact and partial derivatives of $\delta'(s,t)$ are continuous in (s,t,x), it follows that (6.3) holds for all x belonging to some neighbourhood of Y which does not depend on (s,t). Decreasing N_Y we can assume that (6.3) holds on all of N_Y.

Let W be a neighbourhood of Y such that

$$Y \subset W \subset \overline{W} \subset N_Y. \tag{6.4}$$

Take a $\mathcal{C}^\infty$ function $\mu : M \to [0,1]$ such that (i) $\mu = 1$ on Y; (ii) $\mu = 0$ on $M \setminus W$; (iii) μ takes constant values on connected components of level sets of f, so $F(\mu) = 0$.

Now define the map $\hat{\delta} : S \times I \to \mathcal{C}^\infty(M,\mathbb{R})$ by

$$\hat{\delta}(s,t)(x) = \mu(x)\delta'(s,t)(x).$$

Notice that

$$F\bigl(\hat{\delta}(s,t)\bigr) = F\bigl(\mu \cdot \delta'(s,t)\bigr) = \mu F(\delta') + F(\mu)\delta' = \mu F(\delta') > -1.$$

The latter inequality follows from (6.3) and the assumption that $0 \le \mu \le 1$. Hence $\hat{\delta}(S \times I) \subset \Gamma$.

We have to show that $\hat{\delta}$ satisfies conditions (a)-(c) of lemma.

(a) Since $\delta(s,0) = 0$ on Y and E is a linear operator, it follows that $\delta'(s,0) = 0$ on M, and therefore $\hat{\delta}(s,0) = 0$ on M as well.

(b) Since $\operatorname{supp}(\mu) \subset N_Y$, we have that $\operatorname{supp}(\hat{\delta}(s,t)) \subset N_Y$.

(c) Define the map $\hat{\kappa} : S \times I \to \mathcal{C}^\infty(M,M)$ by

$$\hat{\kappa}(s,t)(x) = \mathbf{F}\bigl(x, \hat{\delta}(s,t)(x)\bigr).$$

Then it follows from (6.2) that $\hat{\kappa}(s,t) = \kappa(s,t)$ on Y. Therefore $\eta(s,t) = \hat{\kappa}(s,t)^{-1} \circ \kappa(s,t)$ is fixed on Y. Moreover, since $\hat{\kappa}(s,t)$ preserves leaves of Δ_f, we see that $f \circ \hat{\kappa}(s,t)^{-1} = f$, and therefore

$$f \circ \eta(s,t) = f \circ \hat{\kappa}(s,t)^{-1} \circ \kappa(s,t) = f \circ \kappa(s,t) = \psi(s,t).$$

Case (C). Suppose Y is a regular component of some level set of f, so we can identify N_Y with the product $S^1 \times J$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in the complex plane, and $J = [-1, 1]$ or $[0, 1]$, so that Y corresponds to $S^1 \times 0$. The proof in this case is similar to the proof of Lemma 3.1.

Define a flow $\mathbf{F} : (S^1 \times J) \times \mathbb{R} \to S^1 \times J$ by $\mathbf{F}(z, \tau, \theta) = (ze^{2\pi i\theta}, \tau)$. Consider the universal covering map $q : \mathbb{R} \to S^1$, $q(\theta) = e^{2\pi i\theta}$. Since $\kappa(s, t)$ preserves $Y = S^1 \times 0$, and $\kappa(s, 0) = \mathrm{id}_Y$ for all $s \in S$, the map

$$K : S \times I \times Y \longrightarrow Y = S^1$$

lifts to a function

$$\Delta : S \times I \times Y \longrightarrow \mathbb{R},$$

such that $K = q \circ \Delta$ and $\Delta(s, 0, z) = 0$ for $(s, z) \in S \times Y$. In other words,

$$K(s, t, z) = e^{2\pi i\Delta(s,t,z)}.$$

Since q is a local diffeomorphism, and κ is continuous into $\mathcal{C}^\infty$-topology of $\mathcal{C}^\infty(M, M)$, it follows that the function

$$\delta : S \times I \longrightarrow \mathcal{C}^\infty(Y, \mathbb{R}), \qquad \delta(s, t)(z) = \Delta(s, t, z)$$

is continuous into $\mathcal{C}^\infty$-topology of $\mathcal{C}^\infty(Y, \mathbb{R})$ as well.

Now take a $\mathcal{C}^\infty$ function $\mu : J \to [0, 1]$ satisfying (i) $\mu(0) = 1$, (ii) $\mu = 0$ outside $[-0.5, 0.5] \cap J$, and define the map

$$\hat{\delta} : S \times I \longrightarrow \mathcal{C}^\infty(S^1 \times J, \mathbb{R}) = \mathcal{C}^\infty(N_Y, \mathbb{R})$$

by

$$\hat{\delta}(s, t)(z, \tau) = \mu(\tau)\delta(s, t)(z).$$

Evidently, $\mathrm{supp}\,(\hat{\delta}(s, t)) \subset \mathrm{Int}N_U$. Therefore we can extend $\hat{\delta}(s, t)$ by zero on all of M, and so regard $\hat{\delta}$ as a map $\hat{\delta} : S \times I \to \mathcal{C}^\infty(M, \mathbb{R})$. Similarly to the case (A) one can verify that $\hat{\delta}(S \times I) \subset \Gamma$ and $\hat{\delta}$ has properties (a)-(c). $\square$

Case (B). Suppose Y is a non-orientable connected component of X. Denote

$$\widetilde{X} = \beta^{-1}(X), \qquad \widetilde{N}_{\widetilde{Y}} = \beta^{-1}(N_Y), \qquad \widetilde{Y} = \beta^{-1}(Y).$$

Let also $\widetilde{\mathcal{D}}(\widetilde{M}, \widetilde{X})$ be the subgroup of $\widetilde{\mathcal{D}}(\widetilde{M})$ consisting of diffeomorphisms fixed on $\widetilde{X}$ (and commuting with ξ). Since $\psi(s_0, 0) = \mathrm{id}_M$, it follows that κ lifts to a map $\widetilde{\kappa} : S \times I \to \widetilde{\mathcal{D}}(\widetilde{M})$.

Notice that $\widetilde{Y}$ is a connected orientable surface and $\widetilde{f}$ satisfies axioms (B1)-(B3). Therefore we can apply case (A) of Lemma 6.3 and find a map

$$\hat{\delta} : S \times I \longrightarrow \Gamma = \{\alpha \in \mathcal{C}^\infty(\widetilde{M}, \mathbb{R}) \mid F(\alpha) > -1\}$$

satisfying the conditions (a)-(c): $\hat{\delta}(s, 0) = 0$, $\mathrm{supp}\left(\hat{\delta}(s, t)\right) \subset N_{\hat{Y}}$, and $\hat{\kappa} = \varphi \circ \hat{\delta}(s, t)$ coincides with $\widetilde{\kappa}(s, t)$ on $\hat{Y}$.

Notice that the restriction $\widetilde{\kappa}(s, t)$ to $\widetilde{Y}$ is a lifting of $\kappa(s, t)|_Y$, and therefore it commutes with ξ, that is

$$\widetilde{\kappa}(s, t) \circ \xi(x) = \xi \circ \widetilde{\kappa}(s, t)(x).$$

Moreover, $\widetilde{\kappa}(s, t)|_Y \in \widetilde{\mathcal{S}}_{\mathrm{id}}(\widetilde{f}|_{\widetilde{Y}})$. Then it follows from the construction of Lemma 6.3 that

$$\hat{\delta}(s, t)|_{\widetilde{Y}} \in \widetilde{\Gamma}_{\widetilde{Y}} = \{\alpha \in \mathcal{C}^\infty(\widetilde{Y}, \mathbb{R}) \mid F(\alpha) > -1,\ \alpha \circ \xi = -\alpha\}.$$

In particular we have that

$$\hat{\delta}(s, t) \circ \xi(x) = -\hat{\delta}(s, t)(x), \qquad \forall x \in \widetilde{Y}. \tag{6.5}$$

Now define two maps

$$\hat\delta_1 : S\times I \to \mathcal{C}^\infty(\widetilde{M},\mathbb{R}), \qquad \hat\delta_1(s,t) = \frac{1}{2}\bigl(\hat\delta(s,t) - \hat\delta(s,t)\circ\xi\bigr),$$

$$\hat\kappa_1 = \varphi\circ\hat\delta_1 : S\times I \to \mathcal{C}^\infty(M,M), \qquad \hat\kappa_1(s,t)(x) = \mathbf{F}\bigl(x, \hat\delta_1(s,t)(x)\bigr).$$

Lemma 6.4. *$\hat\delta_1$ and $\hat\kappa_1$ have the following properties*

$$\hat\delta_1(s,t)(x) = \hat\delta_1(s,t)(x), \qquad x\in\widetilde{Y} \tag{6.6}$$

$$\hat\delta_1(s,t)\circ\xi = -\hat\delta_1(s,t), \tag{6.7}$$

$$\hat\kappa_1(s,t)\circ\xi = \xi\circ\hat\kappa_1(s,t), \tag{6.8}$$

$$\hat\delta_1(S\times I)\subset\widetilde{\Gamma}, \tag{6.9}$$

and satisfy conditions (a)-(c) *of Lemma* 6.3*. Hence the map*

$$\widetilde{\eta} : S\times I \longrightarrow \mathcal{D}(M,X), \qquad \widetilde{\eta}(s,t) = \hat\kappa(s,t)_1^{-1}\circ\widetilde{\kappa}(s,t)$$

is a ξ-equivariant lifting of $\kappa(s,t)$ inducing a map $\eta : S\times I\to\mathcal{D}(M,X)$ being a required lifting of ψ.

Proof. (6.7) is evident, (6.6) follows from (6.5). Moreover,

$$\begin{aligned}\hat\kappa_1(s,t)\circ\xi(x) &= \mathbf{F}\bigl(\xi(x), \hat\delta_1(s,t)\circ\xi(x)\bigr) = \mathbf{F}\bigl(\xi(x), -\hat\delta_1(s,t)(x)\bigr)\\ &= \xi\circ\mathbf{F}\bigl(x,\hat\delta_1(s,t)(x)\bigr) = \xi\circ\hat\kappa_1(s,t)(x).\end{aligned}$$

which proves (6.8).

Finally, since $\hat\delta(s,t)\in\Gamma$ we have that $F\bigl(\hat\delta(s,t)\bigr) > -1$ on all of M. Moreover, due to (6.7) and the assumption that $\xi^* F = -F$, it follows that $F\bigl(\hat\delta(s,t)\circ\xi\bigr) = -F\bigl(\hat\delta(s,t)\bigr)$, whence

$$\begin{aligned}F(\hat\delta_1(s,t)) &= \frac{1}{2}F\bigl(\hat\delta(s,t) - \hat\delta(s,t)\circ\xi\bigr) = \frac{1}{2}\Bigl[F\bigl(\hat\delta(s,t)\bigr) + F\bigl(\hat\delta(s,t)\bigr)\Bigr]\\ &= F\bigl(\hat\delta(s,t)\bigr) > -1.\end{aligned}$$

Property (a) is evident, (b) follows from the relation $\xi(\widetilde{N}_{\widetilde{Y}}) = \widetilde{N}_{\widetilde{Y}}$, and (c) from (6.6) and property (c) for $\hat\kappa$. Lemma 6.4 is finished. □

Thus for any connected component Y we can change $\kappa(s,t)$ on N_Y to make it a lifting of $\psi(s,t)$ fixed on Y. Since neighbourhoods N_Y are disjoint, we can make these changes mutually on all of N, and so assume that $\psi(s,t)$ is fixed on X. Then ψ will be the desired lifting η of ψ. Theorem 6.1 is completed. □

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